# Analytical solutions of the lattice Boltzmann BGK model

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**Abstract**

Analytical solutions of the two dimensional triangular and square lattice Boltzmann BGK models have been obtained for the plain Poiseuille flow and the plain Couette flow. The analytical solutions are written in terms of the characteristic velocity of the flow, the single relaxation time $\tau$ and the lattice spacing. The analytic solutions are the exact representation of these two flows without any approximation.

## 1 Introduction

Since the appearance of lattice gas automata (LGA) and its later derivative, the lattice Boltzmann equation method (LBE) as alternative computational methods to study transport phenomena, some analytical solutions have been obtained for these methods for nonuniform flows in 2D or 3D models [1, 2, 3, 4]. They are based on linearized Boltzmann models, and they have employed approximations. In [1, 2, 3], the solution around a global equilibrium with constant density and isotropic velocity (zero velocity) was considered. In [4], the first order and second order of deviation of distribution function from equilibrium were assumed to take a certain form in terms of flow quantities, and the coefficients in this form were obtained using a Chapman-Enskog procedure. These analytic results provided insight for applications of the methods. For example, the analytical results allow one to calculate viscosity from given collision rules and to estimate and to improve bounce-back boundary conditions. They are valuable in enhancing our understanding of the method. Nevertheless, analytical solutions for real flows with boundaries like the Poiseuille flow, which is represented exactly by a second-order finite-difference scheme on a uniform mesh, have not been obtained previously for LGA or LBE. One reason may be that the boundary conditions used in LGA and LBE are not exact. For example, bounce-back or a combination of bounce-back and specular reflection [2] for modeling the non-slip boundary condition are only approximate. The effective non-slip boundary is inside the bounce-back row [2, 4]. Recently, Noble *et al.* [5] have proposed a boundary condition for the lattice Boltzmann BGK model (LBGK) on a triangular lattice. When this boundary condition was applied to plain Poiseuille flow, the steady state solution of the distribution function gave a parabolic velocity profile up to machine accuracy. The result suggests the existence of an analytical solution to LBGK, which is an exact representation of the Poiseuille flow. This analytical solution is derived in this paper together with an analytical solution for the plain Couette flow.

## 2 Analytical Solutions of the triangular lattice LBGK model

First let us consider the lattice Boltzmann model on a triangular lattice. For a channel flow, a triangular lattice is constructed as shown in Fig. 1. There are two types of particles on each node of an FHP model: rest particles (type 0) with $\mathbf{e}_0 = 0$ and moving particles (type 1 to 6) with unit velocity $\mathbf{e}_i = (\cos((i-1)\pi/6), \sin((i-1)\pi/6),\ i = 1, \cdots, 6$ along 6 directions. Consider the particle distribution functions $f_i(\mathbf{x}, t)$, which represent the probability of finding a particle at node $\mathbf{x} = (x, y)$ and time $t$ with velocity $\mathbf{e}_i$. The lattice Boltzmann BGK model proposed in [6, 7] is the equation for the evolution of $f_i$:

$$f_i(\mathbf{x} + \delta\mathbf{e}_i, t + \delta) - f_i(\mathbf{x}, t) = -\frac{1}{\tau}[f_i(\mathbf{x}, t) - f_i^{(0)}(\mathbf{x}, t)], \quad i = 0, \cdots, 6 \tag{1}$$

where $f_i^{(0)}(\mathbf{x}, t)$ is the equilibrium distribution of the particle of type $i$ at $\mathbf{x}, t$, the right hand side represents the collision term and $\tau$ is the single relaxation time which controls the rate of approach to equilibrium. The density per node, $\rho$, and the macroscopic flow velocity, $\mathbf{u}$, are defined in terms of the particle distribution function by

$$\sum_{i=0}^{6} f_i = \rho, \qquad \sum_{i=1}^{6} f_i \mathbf{e}_i = \rho\mathbf{u}. \tag{2}$$

The equilibrium distribution functions depend only on local density and velocity. A suitable equilibrium distribution for the FHP model can be chosen in the following form [7]:

$$\begin{aligned} f_0^{(0)} &= d_0 - \rho u^2 = \alpha\rho - \rho u^2, \\ f_i^{(0)} &= d + \frac{1}{3}\rho[(\mathbf{e}_i \cdot \mathbf{u}) + 2(\mathbf{e}_i \cdot \mathbf{u})^2 - \frac{1}{2}\mathbf{u} \cdot \mathbf{u}], \quad i = 1, \cdots, 6, \end{aligned} \tag{3}$$

where $\alpha$ is an adjustable parameter, $d = \frac{\rho - d_0}{6}$, and $\sum_i f_i^{(0)} = \rho$, $\sum_i f_i^{(0)}\mathbf{e}_i = \rho\mathbf{u}$. Note that Eq. (1) is written in physical units with the value of the lattice link being $\delta$, Using unit speed for particles (with some physical time unit), a time step has the value of $\delta$ as well. A Chapman-Enskog procedure can be applied to Eq. (1) to derive the macroscopic equations of the model. They are given by: the continuity equation (with an error term $O(\delta^2)$ being omitted):

$$\frac{\partial\rho}{\partial t} + \nabla \cdot (\rho\mathbf{u}) = 0, \tag{4}$$

and the momentum equation (with terms of $O(\delta^2)$ and $O(\delta u^3)$ being omitted):

$$\partial_t(\rho u_\alpha) + \partial_\beta(\rho u_\alpha u_\beta) = -\partial_\alpha(c_s^2\rho) + \partial_\beta(2\nu\rho S_{\alpha\beta}), \tag{5}$$

where the Einstein summation convention is used, $S_{\alpha\beta} = \frac{1}{2}(\partial_\alpha u_\beta + \partial_\beta u_\alpha)$ is the strain-rate tensor, the pressure is given by $p = c_s^2\rho$, where $c_s$ is the speed of sound with $c_s^2 = \dfrac{1-\alpha}{2}$, and $\nu = \dfrac{2\tau - 1}{8}\delta$, with $\nu$ being the the kinematic viscosity. The form of the error terms and the derivation of these equations can be found in [8, 9]. The macroscopic equations of LBGK represent the incompressible Navier-Stokes equations in the limit as $\delta \to 0$, $\rho \to \rho_0$ (a constant ) and the Mach number approaches zero,

The plain Poiseuille flow in a channel with width $2L$ and velocity $\mathbf{u} = (u, v)$ is given by:

$$u = u_0(1 - \frac{y^2}{L^2}), \quad v = 0, \quad \frac{\partial p}{\partial x} = -G, \quad \frac{\partial p}{\partial y} = 0, \tag{6}$$

where $G$ is a constant related to the characteristic velocity $u_0$ by

$$G = 2\rho\nu u_0/L^2, \tag{7}$$

and the flow density $\rho$ is a constant. This is an exact solution of the incompressible Navier-Stokes equations:

$$\begin{aligned} &\nabla \cdot \mathbf{u} = 0, \\ &\partial_t(\rho u_\alpha) + \partial_\beta(\rho u_\alpha u_\beta) = -\partial_\alpha p + \mu \partial_{\beta\beta} u_\alpha, \end{aligned} \tag{8}$$

where $\mu = \rho\nu$. Without loss of generality, we assume that $L = 1$ (a simple scaling of $y' = y/L$ makes $y' \in [-1, 1]$). To approximate Poiseuille flow using the lattice Boltzmann model, it is convenient to replace the constant gradient by a body force $\mathbf{g}$ so that $\rho\mathbf{g} = -\nabla p$. The momentum equation of N-S equations with a body force is:

$$\partial_t(\rho u_\alpha) + \partial_\beta(\rho u_\alpha u_\beta) = -\partial_\alpha p + \rho g_\alpha + \mu \partial_{\beta\beta} u_\alpha. \tag{9}$$

The Poiseuille flow can be generated with a body force $\mathbf{g}$ with $\rho g_x = G, g_y = 0$, where the pressure is held constant. An LBGK which incorporates the body force is a modification of Eq. (1) given by:

$$f_i(\mathbf{x} + \delta\mathbf{e}_i, t + \delta) - f_i(\mathbf{x}, t) = -\frac{1}{\tau}[f_i(\mathbf{x}, t) - f_i^{(0)}(\mathbf{x}, t)] + \delta h_i, \tag{10}$$

where the $h_i$ are chosen as:

$$h_0 = 0, \quad h_i = +G/4, \;\; i = 1, 2, 6, \quad h_i = -G/4, \;\; i = 3, 4, 5, \tag{11}$$

so that

$$\sum_i h_i = 0, \;\; \sum_i h_i \mathbf{e}_i = \rho\mathbf{g}, \;\; \sum_i h_i e_{i\alpha} e_{i\beta} = 0.$$

Now suppose there exists a solution $f_i(\mathbf{x}, t)$ of Eq. (10) and it exactly represents the Poiseuille flow. We then expect the following properties:

1. $f_i(\mathbf{x}, t)$ is steady (independent of $t$).
2. $f_i(\mathbf{x}, t)$ is independent of $x$, hence is only a function of $y$, denoted by $f_i(y)$.
3. $f_2(y) = f_6(-y)$, and $f_3(y) = f_5(-y)$ from the symmetry of the flow.
4. $\sum_i f_i(y) = \rho$ (constant).
5. $\sum_i f_i(y) e_{ix} = \rho u(y)$ , where $u(y) = u_0(1 - y^2)$ (remember $L = 1$).
6. $\sum_i f_i(y) e_{iy} = 0$ .

According to Eq. (3), the equilibrium distributions are given by

$$\begin{aligned} f_0^{(0)} &= d_0 - \rho u^2, \quad (u = u(y)) \\ f_1^{(0)} &= d + \frac{\rho}{3}u + \frac{\rho}{2}u^2, \;\; f_4^{(0)} = d - \frac{\rho}{3}u + \frac{\rho}{2}u^2, \\ f_2^{(0)} &= d + \frac{\rho}{6}u, \;\; f_3^{(0)} = d - \frac{\rho}{6}u, \\ f_5^{(0)} &= d - \frac{\rho}{6}u, \;\; f_6^{(0)} = d + \frac{\rho}{6}u. \end{aligned} \tag{12}$$

Using properties 1,2 and Eq. (10) for $i = 0$ gives:

$$f_0(y) = f_0(y) - \frac{1}{\tau}(f_0(y) - f_0^{(0)}(y)),$$

hence

$$f_0(y) = f_0^{(0)}(y) = d_0 - \rho u^2. \tag{13}$$

Eq. (10) for $i = 1$ gives:

$$f_1(y) = f_1(y) - \frac{1}{\tau}(f_1(y) - f_1^{(0)}(y)) + \delta G/4,$$

hence

$$f_1(y) = f_1^{(0)}(y) + \tau\delta G/4. \tag{14}$$

Similarly

$$f_4(y) = f_4^{(0)}(y) - \tau\delta G/4. \tag{15}$$

It is seen that $f_0, f_1, f_2$ are functions of $y^4$, $y^2$ through dependence of $u$ and $u^2$. To find the remaining $f_i(y)$, we note that $f_i^{(0)}, i = 2, 3, 5, 6$ have no $u^2$ term and thus are functions of $y^2$ only, so the following form is suggested:

$$f_i(y) = a_i + b_i y + c_i y^2, \;\; i = 2, 3, 5, 6, \tag{16}$$

where the twelve unknown coefficients $a_i, b_i, c_i$ depend on flow quantities, $\tau$, $dy$, but not on $y$. Using property 3, we obtain

$$a_2 + b_2 y + c_2 y^2 = a_6 - b_6 y + c_6 y^2, \quad a_3 + b_3 y + c_3 y^2 = a_5 - b_5 y + c_5 y^2,$$

which should be true for any $y$, thus:

$$\begin{aligned} a_6 = a_2,\ b_6 = -b_2,\ c_6 = c_2 \\ a_5 = a_3,\ b_5 = -b_3,\ c_5 = c_3. \end{aligned} \tag{17}$$

Similarly, using property 6, we find:

$$2b_2 y + 2b_3 y = 0, \quad \text{hence} \quad b_3 = -b_2. \tag{18}$$

Property 4 with information obtained gives:

$$2(a_2 + a_3) + 2(c_2 + c_3)y^2 + f_0(y) + f_1(y) + f_4(y) = \rho, \tag{19}$$

On using the expressions for $f_0, f_1, f_4$ given in Eq. (13 - 15), we find

$$a_2 + a_3 = \frac{1}{2}(\rho - d_0 - 2d) = 4d \;\; \text{(on using the expression of } d\text{)}, \quad c_3 + c_2 = 0; \tag{20}$$

which gives

$$c_3 = -c_2, \quad a_3 = 2d - a_2. \tag{21}$$

Similarly, property 5 with information obtained yields:

$$a_2 = \frac{1}{6}\rho u_0 + d - \tau\delta G/4; \quad c_2 = -\frac{1}{6}\rho u_0. \tag{22}$$

At this point, only $b_2$ remains unknown. Using Eq. (10) for $i = 2$, we have

$$f_2(y + dy) = f_2(y) - \frac{1}{\tau}(f_2(y) - f_2^{(0)}(y)) + \delta G/4, \tag{23}$$

where $dy$ is the vertical spacing between two lattice rows and $dy = (\sqrt{3}/2)\delta$. On using the expression for $f_2^{(0)}$, we can obtain:

$$c_2[y^2+2ydy+(dy)^2]+b_2y+b_2dy+a_2 = (1-\frac{1}{\tau})(c_2y^2+b_2y+a_2)+\frac{1}{\tau}[d+\frac{1}{6}\rho u_0(1-y^2)]+\delta G/4. \tag{24}$$

The balance of terms linear in $y$ yields:

$$b_2 = -2\tau c_2 dy = \frac{1}{3}\tau\rho u_0 dy, \tag{25}$$

and fortunately the equations for the coefficients of $y^2$ and $y^0$ are both satisfied. It is easy to check that the evolution equations for $f_3, f_5, f_6$ are all satisfied with the choice of $a_i, b_i, c_i$ obtained so far.

Putting these results all together, we find that the quantities

$$\begin{aligned}
f_0 &= d_0 - \rho u^2,\\
f_1 &= d + \frac{\rho}{3}u + \frac{\rho}{2}u^2 + \tau\delta G/4,\\
f_4 &= d - \frac{\rho}{3}u + \frac{\rho}{2}u^2 - \tau\delta G/4,\\
f_2 &= -\frac{1}{6}\rho u_0 y^2 + \frac{1}{3}\tau\rho u_0 y dy + \frac{1}{6}\rho u_0 + d - \tau\delta G/4,\\
f_3 &= +\frac{1}{6}\rho u_0 y^2 - \frac{1}{3}\tau\rho u_0 y dy - \frac{1}{6}\rho u_0 + d + \tau\delta G/4,\\
f_5 &= +\frac{1}{6}\rho u_0 y^2 + \frac{1}{3}\tau\rho u_0 y dy - \frac{1}{6}\rho u_0 + d + \tau\delta G/4,\\
f_6 &= -\frac{1}{6}\rho u_0 y^2 - \frac{1}{3}\tau\rho u_0 y dy + \frac{1}{6}\rho u_0 + d - \tau\delta G/4,
\end{aligned} \tag{26}$$

all satisfy properties 1-6 and that they together with the equilibrium distribution given in Eq. (12) satisfy the LBGK equation, Eq. (9). Hence it is an exact representation of the Poiseuille flow.

Next, let us see to what boundary condition the solution in Eq. (26) corresponds. Taking the bottom boundary with $y = -1, u = 0$, we have

$$\begin{gathered}
f_0 = d_0, \quad f_1 = d + \tau\delta G/4, \quad f_4 = d - \tau\delta G/4\\
f_2 = -\frac{1}{3}\tau\rho u_0 dy + d - \tau\delta G/4, \quad f_3 = +\frac{1}{3}\tau\rho u_0 dy + d + \tau\delta G/4\\
f_5 = -\frac{1}{3}\tau\rho u_0 dy + d + \tau\delta G/4, \quad f_6 = +\frac{1}{3}\tau\rho u_0 dy + d - \tau\delta G/4.
\end{gathered} \tag{27}$$

It is seen that on the bottom, after the collision and forcing, $f_2 = f_5 - 2\tau\delta G/4$, $f_3 = f_6 + 2\tau\delta G/4$. Hence, if a bounce-back boundary condition $f_2 = f_5, f_3 = f_6$, on $f_2, f_3$ is applied at the bottom to replace the collision and forcing step, the error is of order $\delta$.

This shows that the bounce-back boundary condition is first-order accurate. This has been confirmed in computations [9, 10, 11].

To obtain the steady-state analytical solution in the LBGK simulations, the boundary condition should be suitably chosen for the simulation. The boundary condition proposed by Nobel et al. [5] is a suitable choice. If we are looking at a node $B$ on the bottom, after streaming, $f_2$ and $f_3$ are empty at the node $B$ since no particle is coming from outside. Then Eq. (2) with $u = v = 0$ are used to determine $\rho, f_2, f_3$. Then the normal collision with force as given in Eq. (10) is applied to $f_i$ on the boundaries. Suppose that initially, we use uniform density $\rho_0$ and zero velocity through the flow field; then we compute $f_i^{(0)}$ and set $f_i = f_i^{(0)}$ through the field. Since there are no pressure gradients, it is natural that the density at each node remains constant $\rho_0$ (confirmed by simulations). Therefore Eq. (2) can be used to find the unique $f_2, f_3$ with the correct density and velocity, hence it is consistent with the evolution of $f_2, f_3$ in the analytical solution. Simulation results indicate that the numerical solution with this boundary condition approaches the analytical solution as $t \to \infty$.

Next, let us consider a plain Couette flow, where the flow between two parallel plates (corresponding to $y = 0$ and $y = 1$) are driven by the constantly moving top plate with velocity $u_0$. In this case, the solution is given by:

$$u = u_0 y, \;\; 0 \le y \le 1, \quad v = 0, \quad \nabla p = 0, \tag{28}$$

with $\rho$ a constant and with no body force. So the LBGK model Eq. (1) is used. Using a similar procedure, we find the analytical solution of Eq. (1) representing the Couette flow:

$$\begin{aligned} f_0 = d_0 - \rho u^2, \;\; f_1 = d + \frac{\rho}{3}u + \frac{\rho}{2}u^2, \;\; f_4 = d - \frac{\rho}{3}u + \frac{\rho}{2}u^2, \\ f_2 = +\frac{1}{6}\rho u_0 y + d - \frac{1}{6}\tau\rho u_0 dy, \;\; f_3 = -\frac{1}{6}\rho u_0 y + d + \frac{1}{6}\tau\rho u_0 dy, \\ f_5 = -\frac{1}{6}\rho u_0 y + d + \frac{1}{6}\tau\rho u_0 dy, \;\; f_6 = +\frac{1}{6}\rho u_0 y + d - \frac{1}{6}\tau\rho u_0 dy. \end{aligned} \tag{29}$$

We note that these analytical solutions are valid for any $u_0$, $\tau$ and $dy$.

## 3 Analytical Solutions of the square lattice LBGK model

The square lattice Boltzmann BGK model is proven more robust than the triangular model in numerical simulations [9, 12], it is important and interesting to find analytical solutions for it. The square lattice Boltzmann BGK model uses 3 types of particles. Particles of type 1 move along the x axis or the y axis with speed $\mathbf{e}_i = (\cos(\pi(i-1)/2), \sin(\pi(i-1)/2), i = 1, 2, 3, 4$, and particles of type 2 move along the diagonal directions with speed $\mathbf{e}_i = \sqrt{2}(\cos(\pi(i-4-\frac{1}{2})/2), \sin(\pi(i-4-\frac{1}{2})/2), i = 5, 6, 7, 8$. Rest particles of type 0 with $\mathbf{e}_0 = 0$ (speed zero) are also allowed at each node. Each node is connected to its 8 nearest neighbors by 8 links of length $\delta$ (in physical units) or $\sqrt{2}\delta$ as shown in Fig. 2. The single-particle distribution function $f_i(\mathbf{x}, t)$ again satisfies the LBGK model Eq. (1) (with $i = 0, \cdots, 8$). The density $\rho$ and the macroscopic velocoty $\mathbf{u}$ are still defined in Eq. (2). For the square lattice, the equilibrium distribution can be chosen in the following form for

particles of each type (the model d2q9 [7]):

$$f_0^{(0)} = \frac{4}{9}\rho[1 - \frac{3}{2}\mathbf{u}\cdot\mathbf{u}],$$
$$f_i^{(0)} = \frac{1}{9}\rho[1 + 3(\mathbf{e}_i\cdot\mathbf{u}) + \frac{9}{2}(\mathbf{e}_i\cdot\mathbf{u})^2 - \frac{3}{2}\mathbf{u}\cdot\mathbf{u}], \quad i = 1,2,3,4$$
$$f_i^{(0)} = \frac{1}{36}\rho[1 + 3(\mathbf{e}_i\cdot\mathbf{u}) + \frac{9}{2}(\mathbf{e}_i\cdot\mathbf{u})^2 - \frac{3}{2}\mathbf{u}\cdot\mathbf{u}], \quad i = 5,6,7,8, \tag{30}$$

with $\sum_\sigma\sum_i f_{\sigma i}^{(0)} = \rho$, $\sum_\sigma\sum_i f_{\sigma i}^{(0)}\mathbf{e}_{\sigma i} = \rho\mathbf{u}$. The macroscopic equations of the model are the same as given in Eqs. (4,5) with $c_s^2 = 1/3$, and $\nu = \frac{2\tau - 1}{6}\delta$. To incorporate a body force in the model to model Poinseuille flow, Eq. (10) ( $i = 0, \cdots, 8$) is used, with $h_i$ chosen in the following way [13]:

$$h_0 = 0, \quad h_1 = \frac{1}{3}G, \quad h_2 = 0, \quad h_3 = -\frac{1}{3}G, \quad h_4 = 0,$$
$$h_5 = h_8 = \frac{1}{12}G, \quad h_6 = h_7 = -\frac{1}{12}G, \tag{31}$$

To derive an analytical solution of Eq. (10) for the square lattce, we note that the six properties in Section 2 still applies except that property 3 is replaced by:

3. $f_2(y) = f_4(-y)$, $f_5(y) = f_8(-y)$ and $f_6(y) = f_7(-y)$ from the symmetry of the flow.

Using a similar procedure as in Section 2, we can find that

$$f_0(y) = f_0^{(0)}(y) = \frac{4}{9}\rho(1 - \frac{3}{2}u^2), \quad (u = u_0(1 - y^2)), \tag{32}$$

$$f_1(y) = \frac{1}{9}\rho(1 + 3u + 3u^2) + \frac{2}{3}\tau\nu\rho u_0\delta. \tag{33}$$

$$f_3(y) = \frac{1}{9}\rho(1 - 3u + 3u^2) - \frac{2}{3}\tau\nu\rho u_0\delta. \tag{34}$$

and

$$f_i(y) = a_i + b_i y + c_i y^2 + d_i y^3 + e_i y^4, \quad i = 2,4,5,6,7,8, \tag{35}$$

with

$$a_4 = a_2, \ b_4 = -b_2, \ c_4 = c_2, \ d_4 = -d_2, \ e_4 = -e_2,$$
$$a_8 = a_5, \ b_8 = -b_5, \ c_8 = c_5, \ d_8 = -d_5, \ e_8 = -e_5,$$
$$a_7 = a_6, \ b_7 = -b_6, \ c_7 = c_6, \ d_7 = -d_6, \ e_7 = -e_6, \tag{36}$$

and

$$a_2 = -4\tau^4\rho u_0^2\delta^4 + 6\tau^3\rho u_0^2\delta^4 - \frac{7}{3}\tau^2\rho u_0^2\delta^4 + \frac{2}{3}\tau^2\rho u_0^2\delta^2 + \frac{1}{6}\tau\rho u_0^2\delta^4 - \frac{1}{3}\tau\rho u_0^2\delta^2 + \frac{1}{9}\rho - \frac{1}{6}\rho u_0^2,$$
$$b_2 = 4\tau^3\rho u_0^2\delta^3 - 4\tau^2\rho u_0^2\delta^3 + \frac{2}{3}\tau\rho u_0^2\delta^3 - \frac{2}{3}\tau\rho u_0^2\delta,$$
$$c_2 = -2\tau^2\rho u_0^2\delta^2 + \tau\rho u_0^2\delta^2 + \frac{1}{3}\rho u_0^2, \quad d_2 = \frac{2}{3}\tau\rho u_0^2\delta, \quad e_2 = -\frac{1}{6}\rho u_0^2, \tag{37}$$

$$
\begin{aligned}
a_5 = {} & 2\tau^4\rho u_0^2\delta^4 - 3\tau^3\rho u_0^2\delta^4 + \frac{7}{6}\tau^2\rho u_0^2\delta^4 - \frac{1}{3}\tau^2\rho u_0^2\delta^2 - \frac{1}{6}\tau^2\rho u_0\delta^2 - \frac{1}{12}\tau\rho u_0^2\delta^4 + \\
& \frac{1}{6}\tau\rho u_0^2\delta^2 + \frac{1}{12}\tau\rho u_0\delta^2 + \frac{1}{36}\rho + \frac{1}{12}\rho u_0^2 + \frac{1}{12}\rho u_0 + \frac{1}{6}\tau\nu\rho u_0\delta, \\
b_5 = {} & -2\tau^3\rho u_0^2\delta^3 + 2\tau^2\rho u_0^2\delta^3 - \frac{1}{3}\tau\rho u_0^2\delta^3 + \frac{1}{3}\tau\rho u_0^2\delta + \frac{1}{6}\tau\rho u_0\delta, \\
c_5 = {} & \tau^2\rho u_0^2\delta^2 - \frac{1}{2}\tau\rho u_0^2\delta^2 - \frac{1}{6}\rho u_0^2 - \frac{1}{12}\rho u_0, \quad d_5 = -\frac{1}{3}\tau\rho u_0^2\delta, \quad e_5 = \frac{1}{12}\rho u_0^2,
\end{aligned} \tag{38}
$$

and

$$
\begin{aligned}
a_6 = {} & 2\tau^4\rho u_0^2\delta^4 - 3\tau^3\rho u_0^2\delta^4 + \frac{7}{6}\tau^2\rho u_0^2\delta^4 - \frac{1}{3}\tau^2\rho u_0^2\delta^2 + \frac{1}{6}\tau^2\rho u_0\delta^2 - \frac{1}{12}\tau\rho u_0^2\delta^4 + \\
& \frac{1}{6}\tau\rho u_0^2\delta^2 - \frac{1}{12}\tau\rho u_0\delta^2 + \frac{1}{36}\rho + \frac{1}{12}\rho u_0^2 - \frac{1}{12}\rho u_0 - \frac{1}{6}\tau\nu\rho u_0\delta, \\
b_6 = {} & -2\tau^3\rho u_0^2\delta^3 + 2\tau^2\rho u_0^2\delta^3 - \frac{1}{3}\tau\rho u_0^2\delta^3 + \frac{1}{3}\tau\rho u_0^2\delta - \frac{1}{6}\tau\rho u_0\delta, \\
c_6 = {} & \tau^2\rho u_0^2\delta^2 - \frac{1}{2}\tau\rho u_0^2\delta^2 - \frac{1}{6}\rho u_0^2 + \frac{1}{12}\rho u_0, \quad d_6 = -\frac{1}{3}\tau\rho u_0^2\delta, \quad e_6 = \frac{1}{12}\rho u_0^2.
\end{aligned} \tag{39}
$$

Eqs. (32,33,34) together with Eqs. (35,36,37,38,39) completely specify the analytical solution, which is a solution of Eq. (10) and it exactly represents the Poiseuille flow.

Next, let us see to what boundary condition this analytical solution corresponds. Taking the bottom boundary with $y = -1, u = 0$, we find the relation of $f_{\sigma i}$ after the collision and forcing:

$$
\begin{aligned}
f_1 - f_3 &= \frac{4}{3}\tau\nu\rho u_0\delta, \quad f_2 - f_4 = -2\delta^3(4\tau^3\rho u_0^2 - 4\tau^2\rho u_0^2 + \frac{2}{3}\tau\rho u_0^2), \\
f_5 - f_7 &= -\frac{2}{9}\tau^2\rho u_0\delta^2 + \frac{1}{9}\tau\rho u_0^2\delta^2 + 4\tau^3\rho u_0^2\delta^3 - 4\tau^2\rho u_0^2\delta^3 + \frac{2}{3}\tau\rho u_0^2\delta^3, \\
f_6 - f_8 &= +\frac{2}{9}\tau^2\rho u_0\delta^2 - \frac{1}{9}\tau\rho u_0^2\delta^2 + 4\tau^3\rho u_0^2\delta^3 - 4\tau^2\rho u_0^2\delta^3 + \frac{2}{3}\tau\rho u_0^2\delta^3,
\end{aligned} \tag{40}
$$

if a bounce-back boundary condition in which $f_1$ exchanges with $f_3$, $f_2 = f_4$, $f_5 = f_7$, $f_6 = f_8$, is applied at the bottom to replace the collision and forcing step, the error introduced to $f_1$ and $f_3$ is of order $\delta$. This shows that the bounce-back boundary condition is first-order accurate. This has been confirmed in computations [9, 10, 11]. To obtain the steady-state analytical solution derived in this paper in LBGK simulations, the boundary condition should be suitably chosen. No numerical simulation on a square lattice Boltzmann BGK model has obtained an exact solution for the Poiseuille flow so far. Of course, if we use the analytical solution derived as the initial condition and provide the right form of distribution functions on the boundaries, we will be able to obtain the analytical solution in a simulation (confirmed in simulations). Specification of the analytical solution on boundary does not provide a boundary condition of general purpose. Nevertheless, the analytical solution will give some guidance in developing better boundary conditions of general purpose for the model.

The analytical solution for plain Couette flow are given by

$$
f_0 = \frac{4}{9}\rho(1 - \frac{3}{2}u_0^2y^2), \quad f_1 = \frac{1}{9}\rho(1 + 3u_0y + 3u_0^2y^2), \quad f_3 = \frac{1}{9}\rho(1 - 3u_0y + 3u_0^2y^2),
$$

$$
\begin{aligned}
f_2 &= -\frac{1}{3}\tau^2\rho u_0^2\delta^2 + \frac{1}{6}\tau\rho u_0^2\delta^2 + \frac{1}{9}\rho + \frac{1}{3}\tau\rho u_0^2\delta y - \frac{1}{6}\rho u_0^2 y^2, \\
f_4 &= -\frac{1}{3}\tau^2\rho u_0^2\delta^2 + \frac{1}{6}\tau\rho u_0^2\delta^2 + \frac{1}{9}\rho - \frac{1}{3}\tau\rho u_0^2\delta y - \frac{1}{6}\rho u_0^2 y^2, \\
f_5 &= \frac{1}{6}\tau^2\rho u_0^2\delta^2 - \frac{1}{12}\tau\rho u_0^2\delta^2 - \frac{1}{12}\tau\rho u_0\delta + \frac{1}{36}\rho + (-\frac{1}{6}\tau\rho u_0^2\delta + \frac{1}{12}\rho u_0)y + \frac{1}{12}\rho u_0^2 y^2, \\
f_6 &= \frac{1}{6}\tau^2\rho u_0^2\delta^2 - \frac{1}{12}\tau\rho u_0^2\delta^2 + \frac{1}{12}\tau\rho u_0\delta + \frac{1}{36}\rho + (-\frac{1}{6}\tau\rho u_0^2\delta - \frac{1}{12}\rho u_0)y + \frac{1}{12}\rho u_0^2 y^2, \\
f_7 &= \frac{1}{6}\tau^2\rho u_0^2\delta^2 - \frac{1}{12}\tau\rho u_0^2\delta^2 - \frac{1}{12}\tau\rho u_0\delta + \frac{1}{36}\rho + (+\frac{1}{6}\tau\rho u_0^2\delta - \frac{1}{12}\rho u_0)y + \frac{1}{12}\rho u_0^2 y^2, \\
f_8 &= \frac{1}{6}\tau^2\rho u_0^2\delta^2 - \frac{1}{12}\tau\rho u_0^2\delta^2 + \frac{1}{12}\tau\rho u_0\delta + \frac{1}{36}\rho + (+\frac{1}{6}\tau\rho u_0^2\delta + \frac{1}{12}\rho u_0)y + \frac{1}{12}\rho u_0^2 y^2. \qquad (41)
\end{aligned}
$$

For the Couette flow, the top boundary is a moving one, the analytical solution given here will give a guidance in developing a suitable boundary condition for moving boundaries.

We note that these analytical solutions are valid for any $u_0$, $\tau$ and $\delta$. They will enhance our understanding of the method and will give guidance in applications.

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**Figure captions**

Figure 1. The geometry of the plain channel flow.
Figure 2. Schematic of a square lattice.